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# A Generalized Phase-Field System

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The existence, the estimate, and the uniqueness of a solution to a general phase-field system, motivated by Caginalp's model describing the phase changes, are established. The paper extends the results for the already studied types of nonlinearities related to the model. © 1999 Academic Press

## 1. INTRODUCTION, PRELIMINARIES, AND MAIN RESULT

The physical phenomenon of solidification of a liquid is expressed by a system of two parabolic equations (the phase field model) introduced by Caginalp [5]. This model represents a refinement of the classical Stefan problem in two phases by adding a new nonlinear equation used to distinguish between the phases of the material that is involved in the solidification process.

The new mathematical description of the real phenomenon reflects more accurately the physical aspects (superheating, supercooling, the effects of surface tension, separating zone of liquid and solid states, etc). It is natural to try to find a suitable type of nonlinearity capable of revealing the complexity of the physical phenomena, including the phase changes. The first nonlinearity considered was that of Caginalp [5], namely, $F(v) = \frac{1}{2}(v - v^3)$. In this respect other nonlinearities have been proposed, such as those of Bates and Zheng [4], Hoffman and Jiang [9], and Penrose and Fife [14]. The nonlinear equation in the phase field model containing a general nonlinear part has been studied by Moroşanu and Motreanu [12], where the nonlinearity is given in terms of Nemytski's operator. The nonlinear term is (possibly) nonconvex and nonmonotone. Our assumptions cover a large class of nonlinearities, including the known cases as well as new situations.



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In the present paper we prove under general hypotheses for the nonlinear part the existence of a solution of our nonlinear parabolic boundary value problem. In addition, we obtain an estimate of its solution and the uniqueness.

Basic tools in our argument are the Leray–Schauder degree theory, properties of the Nemytsky operator, and the $L_p$-theory of linear parabolic equations.

In the following we describe the framework of our problem. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a $C^2$ boundary $\partial\Omega$ and let $T > 0$. Consider the nonlinear parabolic problem

$(P)$

$$\begin{aligned}
&\frac{\partial u}{\partial t} + l\frac{\partial \phi}{\partial t} = \Delta u + f(x,t) && (x,t) \in Q := \Omega \times (0,T),\\
&\frac{\partial \phi}{\partial t} = \Delta \phi + F(x,t,\phi) + u && (x,t) \in Q,\\
&\frac{\partial u}{\partial \nu} = \frac{\partial \phi}{\partial \nu} = 0 && \text{on } \Sigma := \partial\Omega \times (0,T),\\
&u(x,0) = u_0(x), \quad \phi(x,0) = \phi_0(x) && x \in \Omega,
\end{aligned}$$

where $f \in L^q(Q)$, $q \geq 2$, $u_0 \in W_\infty^2(\Omega)$, $\phi_0 \in W_\infty^2(\Omega)$, with $\frac{\partial u_0}{\partial \nu} = \frac{\partial \phi_0}{\partial \nu} = 0$ on $\partial\Omega$, and $l > 0$ is a constant. The unknowns are the functions $u$ and $\phi$. Precisely, $u = u(x,t)$ denotes the reduced temperature distribution of a material which occupies the region $\Omega$, and $\phi = \phi(x,t)$ represents the phase function used to distinguish between the phases of the material: at the moment $t$ the material is considered to be liquid if $\phi$ is closed to $+1$, while it is considered to be solid if $\phi$ is close to $-1$.

Throughout the paper, by $C$ we will denote various positive constants, for which we will mention the possible dependences.

The following hypotheses for the function $F\colon Q \times \mathbb{R} \to \mathbb{R}$ are assumed:

$(H_1)$ There is a constant $a_0 \in \mathbb{R}$ such that

$$\big(F(x,t,z_1) - F(x,t,z_2)\big)(z_1 - z_2) \leq a_0 (z_1 - z_2)^2,$$
$$\forall (x,t) \in Q, \quad z_1, z_2 \in \mathbb{R}.$$

$(H_2)$ There is a function $\bar{F}\colon Q \times \mathbb{R}^2 \to \mathbb{R}$ fulfilling the relations

$$\big(F(x,t,z_1) - F(x,t,z_2)\big)^2 \leq \bar{F}(x,t,z_1 z_2)(z_1 - z_2)^2,$$
$$\forall (x,t) \in Q, \quad z_1, z_2 \in \mathbb{R},$$
$$|\bar{F}(x,t,z_1,z_2)| \leq c_0\Big(1 + |z_1|^{2r-2} + |z_2|^{2r-2}\Big),$$
$$\forall (x,t) \in Q, \quad z_1, z_2 \in \mathbb{R},$$

for constants $c_0 > 0$ and $r \geq 1$, provided

$$r < \frac{N+2}{N-2} \qquad \text{if } N > 2. \tag{1.1}$$

$(H_3)$ $F(x,t,z) = k(x,t,z) - h(z)$, $\forall (x,t) \in Q$, $z \in \mathbb{R}$, where $k\colon Q \times \mathbb{R} \to \mathbb{R}$ is a (Carathéodory) function with $k(\cdot,\cdot,z)$ measurable on $Q, \forall z \in \mathbb{R}$, $k(x,t,\cdot) \in C(\mathbb{R})$, $\forall (x,t) \in Q$, $k(\cdot,\cdot,0) \in L^\infty(Q)$ and $h \in C^1(\mathbb{R})$ such that

(i) $k(x,t,z)^2 \leq a_1 h(z) z + a_2(1+z^2)$, $\forall (x,t) \in Q, z \in \mathbb{R}$,

(ii) $-b_0 \leq h'(z) \leq b_1(1+|z|^{r-1})$, $\forall z \in \mathbb{R}$,

for positive constants $a_1, a_2, b_0, b_1$.

LEMMA 1.1. (*a*) *Assumption* $(H_2)$ *implies that* $F$ *fulfills the growth condition*

$$|F(x,t,z)| \leq a(1+|z|^r), \qquad \forall (x,t) \in Q, \quad z \in \mathbb{R}. \tag{1.2}$$

(b) *Assumption* $(H_3)$ (i) *implies the relations*

$$k(x,t,z)z \leq \tfrac{1}{2}h(z)z + b_2(1+z^2), \qquad \forall (x,t) \in Q, \quad z \in \mathbb{R}, \tag{1.3}$$

$$h(z)z \geq -b_3 - b_4 z^2, \qquad \forall z \in \mathbb{R}. \tag{1.4}$$

(c) *The functions* $k(x,t,z)$ *and* $h(z)$ *satisfy the growth condition*

$$\max\{|k(x,t,z)|, |h(z)|\} \leq C(1+|z|^r), \qquad \forall (x,t) \in Q, \quad z \in \mathbb{R}. \tag{1.5}$$

*Here* $a, b_2, b_3, b_4, C$ *stand for positive constants*.

*Proof.* (a) Setting $z_1 = z$ and $z_2 = 0$ in $(H_2)$, we get

$$\begin{aligned}|F(x,t,z)| &\leq |F(x,t,0)| + \overline{F}(x,t,z,0)^{1/2}|z| \\ &\leq |F(x,t,0)| + c_0^{1/2}\left(1+|z|^{2r-2}\right)^{1/2}|z|, \qquad \forall z \in \mathbb{R}.\end{aligned}$$

Since $F(\cdot,\cdot,0) \in L^\infty(Q)$, relation (1.2) follows.

(b) Condition $(H_3)$ (i) and Young's inequality show that

$$\begin{aligned}k(x,t,z)z &\leq \frac{1}{2a_1}k(x,t,z)^2 + \frac{a_1}{2}z^2 \\ &\leq \frac{1}{2}h(z)z + b_2(1+z^2), \qquad \forall (x,t) \in Q, \quad z \in \mathbb{R},\end{aligned}$$

so (1.3) is established.

We observe that $(H_3)$ (i) leads to (1.4).

(c) From $(H_3)$ and (1.2) we deduce that

$$\begin{aligned}|h(z)| &\le |F(x,t,z)| + |k(x,t,z)| \\ &\le a(1+|z|^r) + \sqrt{a_1|h(z)z|} + \sqrt{a_2(1+z^2)} \\ &\le \tfrac{1}{2}|h(z)| + C(1+|z|^r).\end{aligned}$$

Then we infer that (1.5) is valid. Q.E.D.

*Remark* 1.1. In the case where $r = 1$, $(H_2)$ implies $(H_1)$. Indeed, by $(H2)$ with $r = 1$, we can write

$$\begin{aligned}&\big(F(x,t,z_1) - F(x,t,z_2)\big)(z_1 - z_2) \\ &\quad\le \tfrac{1}{2}\big(F(x,t,z_1) - F(x,t,z_2)\big)^2 \\ &\quad+ \tfrac{1}{2}(z_1 - z_2)^2 \le \tfrac{1}{2}(3c+1)(z_1 - z_2)^2.\end{aligned}$$

Let us introduce the number

$$\mu = \begin{cases} \infty, & \text{if } N+2-2q < 0, \\ \text{any positive number} \ge qr & \text{if } N+2-2q = 0, \\ \dfrac{q(N+2)}{N+2-2q} & \text{if } N+2-2q > 0. \end{cases} \tag{1.6}$$

Our main result in studying problem $(P)$ is the following

THEOREM 1.1. *Under assumptions* $(H_1)$–$(H_3)$, *problem* $(P)$ *has a unique solution* $(u, \phi) \in W_q^{2,1}(Q) \times W_\mu^{2,1}(Q)$, *where* $\mu$ *is given by* (1.6). *The solution* $(u, \phi)$ *satisfies*

$$\|u\|_{W_q^{2,1}(Q)} + \|\phi\|_{W_\mu^{2,1}(Q)} \le C\big(1 + \|u_0\|_{W_\infty^2(\Omega)} + \|\phi_0\|^r_{W_\infty^2(\Omega)} + \|f\|_{L^q(Q)}\big), \tag{1.7}$$

*where the constant C depends on* $|\Omega|$ (*the measure of* $\Omega$), $T, q, r, a_1, a_2, b_0, c_0$, $\|F(\cdot,\cdot,0)\|_{L^\infty(Q)}$.

*Moreover, given any number* $M > 0$, *if* $(u_1, \phi_1)$, $(u_2, \phi_2)$ *are solutions to* $(P)$ *corresponding to* $f_1, f_2 \in L^q(Q)$, *respectively, for the same initial conditions, such that* $\|\phi_1\|_{L^\mu(Q)}, \|\phi_2\|_{L^\mu(Q)} \le M$, *then the estimate*

$$\|u_1 - u_2\|_{W_q^{2,1}(Q)} + \|\phi_1 - \phi_2\|_{W_\mu^{2,1}(Q)} \le C\|f_1 - f_2\|_{L^q(Q)} \tag{1.8}$$

*holds, where* $C > 0$ *depends on* $|\Omega|$, $T, q, r, a_0, a_1, a_2, M$.

*Remark* 1.2. (i) Simple examples of possible choices for the nonlinearities $k(x,t,z)$ and $h(z)$ show that generally the problem $(P)$ can admit multiple solutions, even infinitely many solutions. Here we are concerned with the existence of a unique solution.

(ii) Taking $N = 3$, $k(x,t,z) = a(x,t)z + b(x,t)z^2$, with $a, b \in L^\infty(Q)$ and $h(z) = z^3$, problem $(P)$ reduces to the parabolic problem $(P_1)$ studied by Hoffman and Jiang [9]. If $N$ is arbitrary, $a(x,t) = \frac{1}{2}$, $b(x,t) = 0$, and $h(z) = \frac{1}{2}z^3$, problem $(P)$ coincides with the nonlinear equation of the Caginalp model (Caginalp [5]). Another nonlinearity already considered is $F(x,t,z) = a_1 z + \cdots + a_{2p-1} z^{2p-1}$, with $a_{2p-1} < 0$ for an arbitrary $N$ (see Elliot and Zheng [6]). A discussion concerning the physical interest in treating nonlinearities $F(x,t,z) = f(z)$, for different functions $f(z)$, can be found in Fife [7], Bates-Fife [3], Bates-Zheng [4], and Penrose and Fife [14].

The rest of the paper is organized as follows. Section 2 concerns the existence and uniqueness of the solution of the nonlinear equation in $(P)$. This topic has been treated in [12, 13], but for the sake of completeness and some new specific arguments in connection with the system structure of problem $(P)$, we include it here. Section 3 contains the proof of Theorem 1.1.

## 2. EXISTENCE AND UNIQUENESS FOR THE NONLINEAR EQUATION OF $(P)$

Consider the nonlinear parabolic boundary value problem

$$(P1)\qquad \begin{cases} \dfrac{\partial v}{\partial t} - \Delta v = F(x,t,v) + g(x,t) & (x,t) \in Q, \\ \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \Sigma, \\ v(x,0) = v_0(x) & x \in \Omega, \end{cases}$$

where $g \in L^q(Q)$, with $q \geq 2$, and $v_0 \in W_\infty^2(\Omega)$ verifying $\frac{\partial v_0}{\partial \nu} = 0$ on $\partial\Omega$.

THEOREM 2.1. *Assume $(H_2)$, $(H_3)$. Problem $(P1)$ admits at least one solution $v \in W_q^{2,1}(Q)$ satisfying the estimate*

$$\|v\|_{W_q^{2,1}(Q)} \leq C\left(1 + \|v_0\|^r_{W_\infty^2(\Omega)} + \|g\|_{L_q(Q)}\right), \tag{2.1}$$

*where $C > 0$ is a constant which depends on $|\Omega|$, $T, q, r, a_1, a_2, b_0, c_0, \|F(\cdot,\cdot,0)\|_{L^\infty(Q)}$.*

*Proof.* To use the Leray–Schauder degree theory we need to choose a suitable space $L^p(Q)$, $p \geq 1$. Namely, we choose $p$ as follows:

$$p = \begin{cases} \text{any number} \geq 2r & \text{if } N = 1 \text{ or } 2, \\ \text{any number in } \left[2r, \dfrac{2(N+2)}{N-2}\right) & \text{if } N > 2. \end{cases} \tag{2.2}$$

Notice that (2.2) makes sense because of (1.1).

Let us define the nonlinear operator $T\colon L^p(Q) \times [0,1] \to L^p(Q)$ as

$$T(w, \lambda) = v = v(w, \lambda), \qquad \forall w \in L^p(Q), \quad \forall \lambda \in [0,1], \tag{2.3}$$

where $v$ is the solution of the linear problem

$$(P2) \qquad \begin{aligned} &\frac{\partial v}{\partial t} - \Delta v = \lambda(k(x,t,w) - h(w) + g(x,t)) && \text{in } Q, \\ &\frac{\partial v}{\partial \nu} = 0 && \text{on } \Sigma, \\ &v(x,0) = v_0(x) && \text{on } \Omega. \end{aligned}$$

We have to check that $T$ is well defined. According to (1.2) we have $F(\cdot,\cdot,w) \in L^{p/r}(Q)$, $\forall w \in L^p(Q)$. Then $F(\cdot,\cdot,w(\cdot)) + g \in L^{\bar{q}}(Q)$, where

$$\bar{q} = \min\left\{q, \frac{p}{r}\right\} \geq 2. \tag{2.4}$$

Using now the $L_p$-theory of parabolic equations (see Ladyzhenskaya, Solonnikov, and Uraltzeva [10, pp. 341–342]), the solution $v$ to problem $(P2)$ exists and is unique with

$$v = v(w, \lambda) \in W_{\bar{q}}^{2,1}(Q). \tag{2.5}$$

The Lions–Peetre imbedding theorem ensures the continuous inclusion

$$W_{\bar{q}}^{2,1}(Q) \subset L^{\bar{\mu}}(Q), \tag{2.6}$$

where

$$\overline{\mu} = \begin{cases} \infty, & \text{if } \dfrac{1}{\overline{q}} - \dfrac{2}{N+2} < 0, \\ \text{any number} \geq p & \text{if } \dfrac{1}{\overline{q}} - \dfrac{2}{N+2} = 0, \\ \left(\dfrac{1}{\overline{q}} - \dfrac{2}{N+2}\right)^{-1} & \text{if} \dfrac{1}{\overline{q}} - \dfrac{2}{N+2} > 0. \end{cases} \tag{2.7}$$

Since $\frac{2(N+2)}{N-2} \leq \frac{\overline{q}(N+2)}{N+2-2\overline{q}}$, we have

$$p < \overline{\mu}. \tag{2.8}$$

By (2.5), (2.6), and (2.8) we derive that $v = v(w, \lambda) \in L^p(Q)$.

We now check the continuity of $T$. Let $w_n \to w$ in $L^p(\Omega)$ and $\lambda_n \to \lambda$ in $[0, 1]$. Denote $v_n^{\lambda n} = T(w_n, \lambda_n), v_n^\lambda = T(w_n, \lambda), v^\lambda = T(w, _\lambda)$. From $(P2)$ and (2.3) we obtain

$$\begin{aligned} &\frac{\partial}{\partial t}\left(v_n^{\lambda_n} - v_n^\lambda\right) - \Delta\left(v_n^{\lambda_n} - v_n^\lambda\right) = (\lambda_n - \lambda)\big(k(x, t, w_n) && \text{in } Q, \\ &\qquad -h(w_n) + g(x,t)\big) && \\ &\frac{\partial}{\partial \nu}\left(v_n^{\lambda_n} - v_n^\lambda\right) = 0 && \text{on } \Sigma, \\ &v_n^{\lambda_n} - v_n^\lambda = 0 && \text{in } \Omega. \end{aligned}$$

The $L_p$-theory provides the estimate

$$\|v_n^{\lambda_n} - v_n^\lambda\|_{W_{\overline{q}}^{2,1}(Q)} \leq C|\lambda_n - \lambda|\Big(\|k(x,t,w_n) - h(w_n)\|_{L^{\overline{q}}(Q)} + \|g\|_{L^{\overline{q}}(Q)}\Big).$$

The sequence $(w_n)$ is bounded in $L^p(Q)$, so $k(\cdot, \cdot, w_n) - h(w_n)$ is bounded in $L^{p/r}(Q)$ according to a well-known property of the Nemytski operator (see Fonseca and Gangbo [8]). By virtue of (2.4), we derive the boundedness of $k(\cdot, \cdot, w_n) - h(w_n)$ in $L^{\overline{q}}(Q)$. Thus, we get $\|v_n^{\lambda_n} - v_n^\lambda\|_{W_{\overline{q}}^{2,1}(Q)} \to 0$ as $n \to \infty$.

Again from $(P2)$ and (2.3) we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(v_n^\lambda - v^\lambda\right) - \Delta\left(v_n^\lambda - v^\lambda\right) = \lambda\big(k(x,t,w_n) && \text{in } Q,\\
&\qquad -h(w_n) - k(x,t,w) + h(w)\big) &&\\
&\frac{\partial}{\partial \nu}\left(v_n^\lambda - v^\lambda\right) = 0 && \text{on } \Sigma,\\
&v_n^\lambda - v^\lambda = 0 && \text{in } \Omega.
\end{aligned}$$

As above, the $L_p$-theory implies the estimate

$$\|v_n^\lambda - v^\lambda\|_{W_{\bar q}^{2,1}(Q)} \le C\left(\|k(x,t,w_n) - h(w_n) - k(x,t,w) + h(w)\|_{L^{\bar q}(Q)}\right).$$

Thus the continuity of the Nemytski operator yields $\|v_n^\lambda - v^\lambda\|_{W_{\bar q}^{2,1}(Q)} \to 0$ as $n \to \infty$.

Now, using the continuous imbedding (2.6) (with (2.8)), we derive the continuity of the mapping $T$ described in (2.3).

Furthermore, the mapping $T$ given by (2.3) is compact. This can be seen by writing it as the composition

$$T\colon L^p(Q) \times [0,1] \to W_{\bar q}^{2,1}(q) \subset L^p(Q),$$

where the second inclusion is compact in view of inequality (2.8) (see Lions [11]).

We show that there exists $\rho > 0$ such that

$$(v,\lambda) \in L^p(Q) \times [0,1], \quad v = T(v,\lambda) \qquad \Rightarrow \|v\|_{L^p(Q)} < \rho. \tag{2.9}$$

Let $v \in L^p(Q)$ solve the problem

$$(P3)\qquad \begin{aligned}
&\frac{\partial v}{\partial t} - \Delta v = \lambda\big(k(x,t,v) - h(v) + g(x,t)\big) && \text{in } Q,\\
&\frac{\partial v}{\partial \nu} = 0 && \text{on } \Sigma,\\
&v(x,0) = v_0(x) && \text{on } \Omega.
\end{aligned}$$

Denote

$$Q_t := \Omega \times (0,t), \qquad t \in (0,T].$$

Multiplying the first equation in $(P3)$ by $v$, integrating over $Q_t$, and using Young's inequality, Green's theorem, and (1.3), we get

$$\begin{aligned}
&\frac{1}{2}\int_\Omega v^2(x,t)\,dx + \int_{Q_t} |\nabla v|^2 dx\,ds + \frac{\lambda}{2}\int_{Q_t} h(v)v\,dx\,ds\\
&\qquad \le C(a_1,a_2)\left(1 + \int_{Q_t} v^2(x,s)\,dx\,ds\right).
\end{aligned}\tag{2.10}$$

By (1.4) and (2.10) we deduce that

$$\tfrac{1}{2}\int_{\Omega} v^2(x,t)\,dx + \int_{Q_t} |\nabla v|^2 dx\,ds \le C(a_1,a_2)\left(1 + \int_{Q_t} v^2(x,s)\,dx\,ds\right);$$

so by Gronwall's inequality we arrive at

$$\tfrac{1}{2}\int_{\Omega} v^2(x,t)\,dx + \int_{Q_t} |\nabla v|^2 dx\,ds \le C(T,a_1,a_2),\ \forall t \in (0,T]. \quad (2.11)$$

Combining with (2.10) we obtain

$$\int_{Q_t} h(v)v\,dx\,ds \le C(a_1,a_2), \quad (2.12)$$

$$\int_{Q_t} v^2(x,t)\,dx\,ds \le C(T,a_1,a_2). \quad (2.13)$$

Multiplying the first equation in $(P3)$ by $\partial v/\partial t$, integrating over $Q_t$, and using Green's theorem and Young's inequality, we see that

$$\begin{aligned}
&\int_{Q_t}\left(\frac{\partial v}{\partial s}\right)^2 dx\,ds + \tfrac{1}{2}\int_{\Omega} |\nabla v(x,t)|^2\,dx \\
&\quad \le C + \tfrac{1}{2}\int_{Q_t}\left(\frac{\partial v}{\partial s}\right)^2 dx\,ds \\
&\qquad + \int_{Q_t} k^2(x,s,v)\,dx\,ds \\
&\qquad - \lambda\int_{\Omega}\int_0^{v(x,t)} h(\tau)\,d\tau\,dx, \qquad \forall t\in(0,T].
\end{aligned}$$

Here we used the differentiation formula

$$\frac{\partial}{\partial s}\int_0^{v(x,s)} h(\tau)\,d\tau = h(v(x,s))\frac{\partial v}{\partial s}(x,s) \qquad \text{for a.e. } s \in (0,T],$$

which can be proved by using the Mean Value Theorem, the continuity of $h$, and the absolute continuity of $v(\cdot,t)$.

Using $(H_3)$ (i) and taking into account (2.11), (2.12), from the inequality above we arrive at the estimate

$$\frac{1}{2}\int_{Q_t}\left(\frac{\partial v}{\partial s}\right)^2 dx\,ds + \frac{1}{2}\int_{\Omega}|\nabla v(x,t)|^2 dx + \lambda\int_{\Omega}\int_0^{v(x,t)} h(\tau)\,d\tau\,dx$$
$$\leq C(T, a_1, a_2), \qquad \forall t \in (0,T]. \tag{2.14}$$

From assumption (H3) (ii), (2.14), and (2.12) we infer that

$$\left\|\frac{\partial v}{\partial t}\right\|_{L^2(Q)} \leq C(T, a_1, a_2, b_0). \tag{2.15}$$

Multiplying now the first equation in $(P3)$ by $\Delta v$, integrating over $Q_t$, and using Green's formula, assumption (H3), Fubini's theorem, and Young's inequality, we can write

$$\frac{1}{2}\int_{\Omega}|\nabla v(x,t)|^2 dx + \int_{Q_t}(\Delta v)^2\,dx\,ds$$
$$= \frac{1}{2}\int_{\Omega}|\nabla v_0|^2 dx - \lambda\int_{Q_t} k(x,s,v)\,\Delta v\,dx\,ds$$
$$+ \lambda\int_{Q_t} h(v)\,\Delta v\,dx\,ds - \lambda\int_{Q_t} g\Delta v\,dx\,ds \leq C + \frac{1}{2}\int_{Q_t}(\Delta v)^2\,dx\,ds$$
$$+ \int_{Q_t} k^2(x,s,v)\,dx\,ds - \lambda\int_{Q_t} h'(v)|\nabla v|^2 dx\,ds, \qquad \forall t \in (0,T].$$

In the computation above we made use of the Green's formula,

$$\int_{Q_t} h(v)\,\Delta v\,dx\,ds = -\int_{Q_t} h'(v)|\nabla v|^2 dx\,ds, \qquad \forall t \in (0,T], \tag{2.17}$$

on the basis of the next chain rule in Lemma 2.1 below.

LEMMA 2.1. $\Delta(h(v(\cdot,t))) = h'(v(\cdot,t))\nabla v(\cdot,t)$, $\forall t \in (0,T]$.

*Proof.* By the Sobolev embedding theorem it is known that $W^{1,\bar{q}}(\Omega) \subset L^{2^*}(\Omega)$ with $2^* = 2N/(N-2)$ if $N > 2$. Thus $v(\cdot,t) \in L^{2^*}(\Omega)$, $\nabla v(\cdot,t) \in (L^{2^*}(\Omega))^N$, $\forall t \in (0,T]$.

Let $t \in (0,T]$, $\varphi \in C_0^\infty(\Omega)$, and $\omega \subset\subset \Omega$ with supp $\varphi \subset \omega$. Using a well-known approximation result (see, e.g., Adams [1] or Brézis [2]), there is a sequence $(v_n) \subset C_0^\infty(\mathbb{R})$ such that $v_n \to v(\cdot,t)$ in $L^{2^*}(\Omega)$, a.e. in $\Omega$, and $\nabla v_n \to \nabla v(\cdot,t)$ in $L^{2^*}(\omega)$ and a.e. in $\omega$. Then for all $n \geq 1$, $1 \leq i \leq N$, the following equality is valid:

$$\int_\omega h(v_n)\frac{\partial \varphi}{\partial x_i}dx = -\int_\omega h'(v_n)\frac{\partial v_n}{\partial x_i}\varphi\, dx. \qquad (*)_n$$

Taking into account hypothesis $(H_3)$ (ii), relation (1.5), and that $h \in C^1(\mathbb{R}, \mathbb{R})$, we can apply the Lebesgue dominated convergence theorem for passing to the limit in $(*)_n$ as $n \to \infty$ to achieve the conclusion of Lemma 2.1. Q.E.D.

*Proof of Theorem* 2.1 (*continued*). Using assumption $(H_3)$ (i), we can express (2.16) as follows:

$$\begin{aligned}\tfrac{1}{2}\int_\Omega |\nabla v(x,t)|^2 dx + \tfrac{1}{2}\int_{Q_t}(\Delta v^2)\,dx\,ds + \lambda\int_{Q_t} h'(v)|\nabla v|^2 dx\,ds \\ \le C + a_1\int_{Q_t} h(v)v\,dx\,ds + a_2\int_{Q_t} v^2 dx\,ds, \qquad \forall t \in (0,T].\end{aligned}$$

By (2.12), (2.13) we see that

$$\begin{aligned}\tfrac{1}{2}\int_\Omega |\nabla v(x,t)|^2 dx + \tfrac{1}{2}\int_{Q_t}(\Delta v)^2 dx\,ds + \lambda\int_{Q_t} h'(v)|\nabla v|^2 dx\,ds \\ \le C(T, a_1, a_2), \qquad \forall t \in (0,T].\end{aligned} \tag{2.18}$$

From (2.18), assumption $(H_3)$ (ii), and (2.11) we conclude that

$$\|\Delta v\|_{L^2(Q)} \le C(T, a_1, a_2, b_0). \tag{2.19}$$

We handle now problem $(P3)$ and estimates (2.15), (2.19). Applying the $L_p$-theory (for $p = 2$) to the parabolic problem $(P3)$, we see that

$$\begin{aligned}\|v\|_{W_2^{2,1}(Q)} &\le C\big(\|v_0\|_{W_\infty^2(\Omega)} + \|k(\cdot,\cdot,v) - h(v) + g\|_{L^2(Q)}\big) \\ &= C\left(\|v_0\|_{W_\infty^2(\Omega)} + \left\|\frac{\partial v}{\partial t} - \Delta v\right\|_{L^2(Q)}\right).\end{aligned} \tag{2.20}$$

Taking into account (2.15), (2.19), it is clear that (2.20) yields

$$\|v\|_{W_2^{2,1}(Q)} \le C(|\Omega|, T, a_1, a_2, b_0). \tag{2.21}$$

Notice that the Lions–Peetre imbedding theorem (see Lions [11]) ensures the continuous inclusion $W_2^{2,1}(Q) \subset L^{\tilde{\mu}}(Q)$, where

$$\tilde{\mu} = \begin{cases} \infty, & \text{if } N = 1, \\ \text{any positive number} & \text{if } N = 2, \\ \dfrac{2(N+2)}{N-2} & \text{if } N > 2. \end{cases}$$

Since (cf. (2.2)) for $N > 2$, $\tilde{\mu} = \frac{2(N+2)}{N-2} > p$, one has the continuous imbedding

$$W_2^{2,1}(Q) \subset L^p(Q). \tag{2.22}$$

In conjunction with (2.21), this leads to the conclusion that the claim in (2.9) holds true. Denoting

$$B_\rho := \{v \in L^p(Q) : \|v\|_{L^p(Q)} < \rho\},$$

(2.9) ensures that

$$T(v, \lambda) \neq v, \qquad \forall v \in \partial B_\rho, \quad \forall \lambda \in [0,1]. \tag{2.23}$$

According to property (2.23) and the compactness of $T(\cdot, \lambda)$: $L^p(Q) \to L^p(Q)$, we may consider the Leray–Schauder degree,

$$\deg\big(\mathrm{Id}_{L^p(Q)} - T_\lambda, B_\rho, 0\big), \qquad \forall \lambda \in [0,1]$$

(see Schwartz [15]). The homotopy invariance of the Leray–Schauder degree enables us to write the equality

$$\deg\big(\mathrm{Id}_{L^p(Q)} - T(\cdot,0), B_\rho, 0\big) = \deg\big(\mathrm{Id}_{L^p(Q)} - T(\cdot,1), B_\rho, 0\big). \tag{2.24}$$

Choose $\rho > 0$ large enough so that the ball $B_\rho$ contains the unique solution of the linear heat equation $v - T(v,0) = 0$. It follows that

$$\deg\big(\mathrm{Id}_{L^p(Q)} - T(\cdot,0), B_\rho, 0\big) = 1. \tag{2.25}$$

From (2.24), (2.25) we conclude that problem $(P_1)$ has a solution $v \in W_{\bar{q}}^{2,1}(Q)$.

We have to show that

$$v \in W_q^{2,1}(Q). \tag{2.26}$$

If $q \leq \frac{p}{r}$, then $\bar{q} = q$ and (2.26) is valid. If $q > \frac{p}{r}$ (equivalently, in view of (2.4), $\bar{q} = \frac{p}{r}$), we use a bootstrap argument. By the $L_p$-theory and (2.6) we know that $v \in W^{2,1}_{p/r}(Q) \subset L^{\overline{\mu}}$ with $\overline{\mu} > p$ (cf. (2.8)). So $q \geq \frac{\mu}{r} \geq \frac{p}{r}$. Now, we repeat the reasoning with $v \in L^{\overline{\mu}}(Q)$ ($\overline{\mu}$ in place of $p$). After finitely many steps the condition $\frac{p}{r} \geq q$ is achieved, which proves (2.26).

It remains to establish the estimate (2.1). Toward this end we point out that the preceding discussion permits us to proceed under the assumption $\frac{p}{r} \geq q \geq 2$. The estimates (2.11), (2.14), (2.18) (written with fixed given functions $v_0$ and $g$) show that (2.21) in conjunction with (2.22) takes the form

$$\|v\|_{L^p(Q)} \leq C\big(|\Omega|, T, a_1, a_2, b_0, q, r\big)\big(1 + \|v_0\|_{W^2_\infty(Q)} + \|g\|_{L^q(Q)}\big). \quad (2.27)$$

Then the $L_p$-theory applied to (P1), together with (1.2) and (2.27), enables us to write for any solution $v \in W^{2,1}_q(Q)$ that

$$\begin{aligned}\|v\|_{W^{2,1}_q(Q)} &\leq C\big(|\Omega|, T, a_1, a_2, b_0, q, r\big) \\ &\qquad \times\big(\|v_0\|_{W^2_\infty(\Omega)} + \|F(x,t,v)\|_{L^q(Q)} + \|g\|_{L^q(Q)}\big) \\ &\leq C\big(|\Omega|, T, a_1, a_2, b_0, q, r\big)\Big(\|v_0\|_{W^2_\infty(\Omega)} \\ &\qquad + C\big(c_0, \|F(\cdot,\cdot,0)\|_{L^\infty(Q)}\big)\big(1 + \|v\|^r_{L_p(Q)}\big) + \|g\|_{L^g(Q)}\Big),\end{aligned}$$

which leads to (2.1). Q.E.D.

THEOREM 2.2. *Assume* $(H_1)$, $(H_2)$ *and let* $M > 0$. *If* $g_1(x,t)$, $g_2(x,t)$ *belong to* $L^q(Q)$, $q \geq 2$, *and* $v_1(x,t)$, $v_2(x,t) \in W^{2,1}_q(Q)$ *are corresponding solutions to problem* $(P1)$ *with* $\|v_1\|_{L^\mu(Q)}, \|v_2\|_{L^\mu(Q)} \leq M$, *then the estimate*

$$\|v_1 - v_2\|_{W^{2,1}_q(Q)} \leq C\|g_1 - g_2\|_{L^q(Q)} \quad (2.28)$$

*holds, where* $C > 0$ *is a constant which depends on* $|\Omega|, T, q, r, a_0, M$. *In particular, the uniqueness of the solution to* $(P1)$ *holds.*

*Proof.* It is known that $v_1 - v_2 \in W^{2,1}_q(Q)$ satisfies

$(P4)$

$$\begin{cases} \dfrac{\partial}{\partial t}(v_1 - v_2) - \Delta(v_1 - v_2) = F(x,t,v_1) - F(x,t,v_2) & \\ \qquad\qquad +g_1(x,t) - q_2(x,t) & (x,t) \in Q, \\ \dfrac{\partial}{\partial \nu}(v_1 - v_2) = 0 & \text{on } \Sigma, \\ (v_1 - v_2)(x,0) = 0 & x \in \Omega. \end{cases}$$

Multiplying the first equality in $(P4)$ by $v_1 - v_2$, integrating over $Q_t$, $t \in (0,T]$, and using Green's formula and the Cauchy–Schwarz inequality, we obtain

$$\frac{1}{2}\int_\Omega (v_1(x,t) - v_2(x,t))^2 dx + \int_{Q_t} |\nabla(v_1 - v_2)|^2 dx\,ds$$
$$\leq \int_{Q_t} (F(x,t,v_1) - F(x,t,v_2))(v_1 - v_2)\,dx\,ds + \frac{1}{2}\int_{Q_t} (g_1 - g_2)^2 dx\,ds$$
$$+ \frac{1}{2}\int_{Q_t} (v_1 - v_2)^2 dx\,ds \qquad \forall (x,t) \in Q.$$

Because of the assumption $(H_1)$ and by means of Gronwall's inequality, it results that

$$\|v_1 - v_2\|^2_{L^2(Q)} \leq C(|\Omega|, T, a_0)\|g_1 - g_2\|^2_{L^q(Q)}. \tag{2.29}$$

According to the definition of $\mu$ (see (1.1) and (2.8)), it is permitted to admit that $\mu \geq qr$. This guarantees that $v_1 - v_2 \in L^\mu(Q) \subseteq L^{qr}(Q)$, which in conjunction with relation (1.2) yields that $F(\cdot,\cdot,v_1) - F(\cdot,\cdot,v_2) \in L^q(Q)$. Now we may apply the $L_p$-theory, which gives the estimate

$$\|v_1 - v_2\|^2_{W_q^{2,1}(Q)} \leq C(|\Omega|, T, N)\Big(\|F(x,t,v_1) - F(x,t,v_2)\|^2_{L^q(Q)}$$
$$+ \|g_1 - g_2\|^2_{L^q(Q)}\Big). \tag{2.30}$$

The inequality $\mu \geq qr$ allows us to fix a number $p$ such that

$$2 \leq q \leq \frac{\mu q}{\mu + q - qr} \leq p \leq \mu. \tag{2.31}$$

Consequently, the next sequence of imbeddings holds:

$$W_q^{2,1}(Q) \subset L^\mu(Q) \subset L^p(Q) \subset L^q(Q) \subset L^2(Q). \tag{2.32}$$

From $(H_2)$, (2.31) and Hölder's inequality it is seen that

$$\|F(x,t,v_1) - F(x,t,v_2)\|_{L^q(Q)}$$
$$\leq \|\overline{F}(x,t,v_1,v_2)^{1/2}|v_1 - v_2|\,\|_{L^q(Q)}$$
$$= \left(\int_Q \overline{F}(x,t,v_1,v_2)^{q/2}|v_1 - v_2|^q dx\,dt\right)^{1/q}$$
$$\leq \left(\int_Q \overline{F}(x,t,v_1,v_2)^{m/2} dx\,dt\right)^{2/m\cdot 1/2} \|v_1 - v_2\|_{L^p(Q)}, \tag{2.33}$$

where we denoted $m := pq/(p-q)$. The computation above makes sense because $\overline{F}(x,t,v_1,v_2) \in L^{m/2}(Q)$. Indeed, taking into account the growth condition in $(H_2)$, $\overline{F}(x,t,v_1,v_2) \in L^{\mu/2(r-1)}(Q)$ whenever $v_1, v_2 \in L^{\mu}(Q)$, and by (2.31), it is true that

$$\frac{\mu}{r-1} \geq m > 2. \tag{2.34}$$

Combining (2.30), (2.32) we arrive at

$$\|v_1 - v_2\|^2_{W_q^{2,1}(Q)} \leq C(|\Omega|, T, N)\Big(\|\overline{F}(x,t,v_1,v_2)\|_{L^{m/2}(Q)} \cdot \|v_1 - v_2\|^2_{L^p(Q)} + \|g_1 - g_2\|^2_{L^q(Q)}\Big). \tag{2.35}$$

By virtue of $(H_2)$, (2.34), and (2.35) we obtain that

$$\|v_1 - v_2\|^2_{W_q^{2,1}(Q)} \leq C(|\Omega|, T, r, q, N)(1 + 2M^{2(r-1)}) \times \Big(\|v_1 - v_2\|^2_{L^p(Q)} + \|g_1 - g_2\|^2_{L^q(Q)}\Big). \tag{2.36}$$

By the sequence of imbeddings

$$W_q^{2,1}(Q) \subset L^{\mu}(Q) \subset L^p(Q) \subset L^q(Q) \subset L^2(Q),$$

the interpolation inequality yields that $\forall \varepsilon > 0\, \exists C(\varepsilon) > 0$ such that

$$\|v\|_{L^p(Q)} \leq \varepsilon \|v\|_{W_q^{2,1}(Q)} + C(\varepsilon)\|v|_{L^2(Q)}, \qquad \forall v \in W_q^{2,1}(Q). \tag{2.37}$$

From (2.36), (2.37), and (2.29) we derive that

$$\begin{aligned}(1 - \varepsilon C(|\Omega|, T, r, q, N, M))\|v_1 - v_2\|^2_{W_q^{2,1}(Q)} &\leq C(|\Omega|, T, r, q, N, M)\Big(C(\varepsilon)C(|\Omega|, T, a_0)\|g_1 - g_2\|^2_{L^q(Q)} \\ &\quad + \|g_1 - g_2\|^2_{L^q(Q)}\Big).\end{aligned} \tag{2.38}$$

For $\varepsilon > 0$ with $1 - \varepsilon C(|\Omega|, T, r, q, N, M) > 0$, (2.38) implies estimate (2.28). Q.E.D.

## 3. PROOF OF THEOREM 1.1

### 3.1. *A Priori Estimates*

Consider a positive integer $p$ satisfying (2.2). This is possible because of requirement (1.1). We introduce the homotopy $H$: $L^q(Q) \times L^p(Q) \times [0,1] \to L^q(Q) \times L^p(Q)$ as

$$H(v, \psi, \lambda) = (u, \phi), \qquad \forall (v, \psi, \lambda) \in L^q(Q) \times L^p(Q) \times [0,1], \tag{3.1}$$

where $(u, \phi)$ is the unique solution of the problem

$$
(P5) \qquad \begin{aligned}
&\frac{\partial u}{\partial t} - \Delta u = \lambda\left(-l\frac{\partial \phi}{\partial t} + f(x,t)\right) && (x,t) \in Q,\\
&\frac{\partial \phi}{\partial t} - \Delta \phi - F(x,t,\phi) = \lambda v && (x,t) \in Q,\\
&\frac{\partial u}{\partial \nu} = \frac{\partial \phi}{\partial \nu} = 0 && \text{on } \Sigma,\\
&u(x,0) = u_0(x), \quad \phi(x,0) = \phi_0(x) && x \in \Omega.
\end{aligned}
$$

We check that the homotopy $H$ is well defined. To this end, for each $\lambda \in [0,1]$ and $v \in L^q(Q)$, we focus on the nonlinear parabolic equation in $(P5)$:

$$
(E_{\lambda, v}) \qquad \begin{cases}
\dfrac{\partial \phi}{\partial t} - \Delta \phi - F(x,t,\phi) = \lambda v & (x,t) \in Q,\\
\dfrac{\partial \phi}{\partial \nu} = 0 & \text{on } \Sigma,\\
\phi(x,0) = \phi_0(x) & x \in \Omega.
\end{cases}
$$

Appling Theorem 2.1 with $g = \lambda v \in L^q(Q)$, on the basis of the growth condition for $F(x,t,z)$ in Lemma 1.1, we know that problem $(E_{\lambda, v})$ has a solution $\phi \in W_q^{2,1}(Q)$. Furthermore, this is unique according to Theorem 2.2. The imbedding theorem of Lions–Peetre [11] ensures the continuous inclusion $W_q^{2,1}(Q) \subset W_2^{2,1}(Q) \subset L^p(Q)$ (cf. (2.22)). Then, with this $\phi \in L^p(Q)$, the first (linear) parabolic equation in $(P5)$ has a unique solution $u \in W_q^{2,1}(Q) \subset L^q(Q)$. Hence the nonlinear mapping $H$ introduced in (3.1) is well defined.

LEMMA 3.1. *The mapping* $H$: $L^q(Q) \times L^p(Q) \times [0,1] \to L^q(Q) \times L^p(Q)$ *in* (3.1) *has the following properties*:

(i) $H(\cdot, \cdot, \lambda)$: $L^q(Q) \times L^p(Q) \to L^q(Q) \times L^p(Q)$ *is compact for every* $\lambda \in [0,1]$, *i.e., it is continuous and maps bounded sets into relatively compact sets.*

(ii) *For every* $\varepsilon > 0$ *and every bounded set* $A \subset L^q(Q) \times L^p(Q)$ *there exists* $\delta > 0$ *such that*

$$\|H(v, \psi, \lambda_1) - H(v, \psi, \lambda_2)\|_{L^q(Q) \times L^p(Q)} < \varepsilon$$

*whenever* $(v, \psi) \in A$ *and* $|\lambda_1 - \lambda_2| < \delta$.

*Proof.* (i) Let us check the continuity of $H(\cdot,\cdot,\lambda)$ at the point $(\bar{v},\bar{\psi}) \in L^q(Q) \times L^p(Q)$. Let $(\bar{u},\bar{\phi}) = H(\bar{v},\bar{\psi},\lambda)$, and for any $(v,\psi) \in L^q(Q) \times L^p(Q)$, $(u,\phi) = H(v,\psi,\lambda)$. By (3.1) and $(P5)$ we derive

$$\begin{aligned}
&\frac{\partial}{\partial t}(u-\bar{u}) - \Delta(u-\bar{u}) = -\lambda l \frac{\partial}{\partial t}(\phi-\bar{\phi}) && \text{in } Q,\\
&\frac{\partial}{\partial t}(\phi-\bar{\phi}) - \Delta(\phi-\bar{\phi}) = F(x,t,\phi) - F(x,t,\bar{\phi}) + \lambda(v-\bar{v}) && \text{in } Q,\\
&\frac{\partial}{\partial \nu}(u-\bar{u}) = \frac{\partial}{\partial \nu}(\phi-\bar{\phi}) = 0 && \text{on } \Sigma,\\
&(u-\bar{u})(x,0) = 0, \quad (\phi-\bar{\phi})(x,0) = 0 && x \in \Omega.
\end{aligned} \tag{3.2}$$

Theorem 2.2 applied to the second equation in (3.2) ensures that

$$\|\phi-\bar{\phi}\|_{W_q^{2,1}(Q)} \le C\|v-\bar{v}\|_{L^q(Q)}. \tag{3.3}$$

Applying the $L_p$-theory to the first equation in (3.2), it follows that there is a constant $C > 0$ such that

$$\|u-\bar{u}\|_{W_q^{2,1}(Q)} \le C\left\|\frac{\partial}{\partial t}(\phi-\bar{\phi})\right\|_{L^q(Q)}.$$

This and (3.3) yield a new constant $C > 0$ such that

$$\|u-\bar{u}\|_{L^q(Q)} \le C\|v-\bar{v}\|_{L^q(Q)}. \tag{3.4}$$

Relations (3.3), (3.4) ensure that the continuity of the map $H(\cdot,\cdot,\lambda)$ at $(\bar{u},\bar{\phi})$ is true.

Since $H(\cdot,\cdot,\lambda)$ can be expressed as the composition

$$L^q(Q) \times L^p(Q) \to W_q^{2,1}(Q) \times W_q^{2,1}(Q) \subset L^q(Q) \times L^p(Q), \tag{3.5}$$

where the second map is a compact inclusion, by the Lions–Peetre imbedding theorem [11] the map $H(\cdot,\cdot,\lambda)$ is compact.

(ii) Fix $\varepsilon > 0$ and a bounded set $A \subset L^q(Q) \times L^p(Q)$. Consider $(u_1,\phi_1,\lambda_1), (u_2,\phi_2,\lambda_2) \in L^q(Q) \times L^p(Q) \times [0,1]$ solving $(P5)$, where we take any $v \in A_1 := \mathrm{pr}_1(A)$ (the projection of $A$ onto the first factor). Then we have

$$
\begin{aligned}
&\frac{\partial}{\partial t}(u_1 - u_2) - \Delta(u_1 - u_2) = -l\left(\lambda_1 \frac{\partial \phi_1}{\partial t} - \lambda_2 \frac{\partial \phi_2}{\partial t}\right) && \text{in } Q,\\
&\qquad\qquad + (\lambda_1 - \lambda_2) f\\
&\frac{\partial}{\partial t}(\phi_1 - \phi_2) - \Delta(\phi_1 - \phi_2) = F(x,t,\phi_1) - F(x,t,\phi_2) && \text{in } Q,\\
&\qquad\qquad + (\lambda_1 - \lambda_2) v\\
&\frac{\partial}{\partial \nu}(u_1 - u_2) = \frac{\partial}{\partial \nu}(\phi_1 - \phi_2) = 0 && \text{on } \Sigma,\\
&(u_1 - u_2)(x,0) = 0, \quad (\phi_1 - \phi_2)(x,0) = 0 && x \in \Omega.
\end{aligned}
\tag{3.6}
$$

Theorem 2.2 applied to the second equation in $(P5)$ (with $g_1 = \lambda_1 v$, $g_2 = \lambda_2 v$) ensures that

$$\|\phi_1 - \phi_2\|_{W_q^{2,1}(Q)} \le C|\lambda_1 - \lambda_2|\, \|v\|_{L^q(Q)} \le C(A_1)|\lambda_1 - \lambda_2|. \tag{3.7}$$

By the $L_p$-theory applied to the first equation in (3.6) we obtain

$$
\begin{aligned}
&\|u_1 - u_2\|_{W_q^{2,1}(Q)}\\
&\qquad \le C\left[|\lambda_1 - \lambda_2|\left(\left\|\frac{\partial \phi_1}{\partial t}\right\|_{L^q(Q)} + \|f\|_{L^q(Q)}\right) + \lambda_2\left\|\frac{\partial \phi_1}{\partial t} - \frac{\partial \phi_2}{\partial t}\right\|_{L^q(Q)}\right].
\end{aligned}
$$

Estimate (2.1) ensures that $\|\partial \phi_1 / \partial t\|_{L^q(Q)}$ is bounded because $A_1$ is bounded. Then, on the basis of (3.7), the estimate above becomes

$$\|u_1 - u_2\|_{W_q^{2,1}(Q)} \le C(A_1)|\lambda_1 - \lambda_2|. \tag{3.8}$$

We see from (3.7) and (3.8) that assertion (ii) is verified. Q.E.D.

LEMMA 3.2. *Under assumptions $(H_1)$–$(H_3)$ there exists a number $\rho > 0$ with the property*

$$H(u, \phi, \lambda) = (u, \phi) \Rightarrow \|(u, \phi)\|_{L^q(Q) \times L^p(Q)} < \rho. \tag{3.9}$$

*Proof.* Using (3.1), the equality $H(u, \phi, \lambda) = (u, \phi)$ is equivalent to

$$(P6)\quad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = \lambda\left(-l\dfrac{\partial \phi}{\partial t} + f(x,t)\right) & \text{in } Q, \\ \dfrac{\partial \phi}{\partial t} - \Delta\phi - F(x,t,\phi) = \lambda u & \text{in } Q, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial \phi}{\partial \nu} = 0 & \text{on } \Sigma, \\ u(x,0) = u_0(x), \quad \phi(x,0) = \phi_0(x) & x \in \Omega. \end{cases}$$

Multiplying the first equation in $(P6)$ by $u + \lambda l\phi$, integrating over $Q_t$, $(t \in (0,T])$, and using Fubini's theorem, Green's formula, and Young's inequality leads to

$$\frac{1}{2}\int_\Omega (u(x,t) + \lambda l\phi(x,t))^2 dx + \int_{Q_t} \nabla u(\nabla u + \lambda l \nabla\phi)\, dx\, ds$$
$$\leq C\left(1 + \int_{Q_t}(u^2(x,s) + \phi^2(x,s))dx\, ds\right), \qquad \forall t \in (0,T]. \quad (3.10)$$

Multiplying the second equation in $(P6)$ by $\phi$, integrating over $Q_t(t \in (0,T])$, and using Green's formula and assumption $(H_3)$, it turns out that

$$\frac{1}{2}\int_\Omega \phi(x,t)^2 dx + \int_{Q_t} |\nabla\phi|^2 dx\, ds$$
$$= \frac{1}{2}\int_\Omega \phi_0^2 dx + \int_{Q_t}(k(x,s,\phi) - h(\phi))\phi\, dx\, ds$$
$$+ \lambda\int_{Q_t} u\phi\, dx\, ds, \qquad \forall t \in (0,T].$$

Relation (1.3) and Young's inequality show that

$$\frac{1}{2}\int_\Omega \phi(x,t)^2 dx + \int_{Q_t} |\nabla\phi|^2 dx\, ds + \frac{1}{2}\int_{Q_t} h(\phi)\phi\, dx\, ds$$
$$\leq C(a_1, a_2)\left(1 + \int_{Q_t}(u^2(x,s) + \phi^2(x,s))dx\, ds\right)$$
$$\forall t \in (0,T], \quad (3.11)$$

while relation (1.4) ensures that

$$\frac{1}{2}\int_\Omega \phi(x,t)^2 dx + \int_{Q_t} |\nabla\phi|^2 dx\, ds \le C(a_1,a_2)\left(1 + \int_{Q_t} \big(u^2(x,s) + \phi^2(x,s)\big) dx\, ds\right), \quad \forall t \in (0,T]. \tag{3.12}$$

Adding (3.10) and (3.12) multiplied by $\frac{1}{2} + \lambda^2 l^2$, we see that

$$\frac{1}{2}\int_\Omega \Big[\big(u(x,t) + \lambda l \phi(x,t)\big)^2 + \big(\tfrac{1}{2} + \lambda^2 l^2\big)\phi(x,t)^2\Big] dx + \int_{Q_t} \Big[\nabla u(\nabla u + \lambda l \nabla\phi) + \big(\tfrac{1}{2} + \lambda^2 l^2\big)|\nabla\phi|^2\Big] dx\, ds \le C(a_1,a_2) \left(1 + \int_{Q_t} \big(u^2(x,s) + \phi^2(x,s)\big) dx\, ds\right), \qquad \forall t \in (0,T]. \tag{3.13}$$

By Gronwall's lemma, we derive from (3.13) the estimate

$$\int_\Omega \big(u^2(x,t) + \phi^2(x,t)\big) dx \le C(T,a_1,a_2), \qquad \forall t \in (0,T],$$

which substituted in (3.13) yields the existence of a constant $C(T, a_1, a_2) > 0$ such that

$$\int_\Omega \big(u^2(x,t) + \phi^2(x,t)\big) dx + \int_{Q_t} \big(|\nabla u|^2 + |\nabla\phi|^2\big) dx\, ds \le C(T,a_1,a_2), \quad \forall t \in (0,T]. \tag{3.14}$$

Combining (3.11) and (3.14), it turns out that, for a constant $C(T, a_1, a_2) > 0$, one has

$$\int_\Omega \phi(x,t)^2 dx + \int_{Q_t} |\nabla\phi|^2 dx\, ds + \int_{Q_t} h(\phi)\phi\, dx\, ds \le C(T,a_1,a_2), \quad \forall t \in (0,T]. \tag{3.15}$$

If we now multiply the first equation in $(P6)$ by $\Delta(u + \lambda l \phi)$ and integrate over $Q_t$, ($t \in (0,T]$), by Green's formula and Fubini's theorem it is seen that

$$\frac{1}{2}\int_\Omega |\nabla u + \lambda l \nabla \phi|^2(x,t)\,dx + \int_{Q_t} (\Delta u)^2\,dx\,ds + \lambda l \int_{Q_t} \Delta u \Delta \phi\,dx\,ds$$

$$= \frac{1}{2}\int_\Omega |\nabla u_0 + \lambda l \nabla \phi_0|^2 dx - \lambda \int_{Q_t} f\Delta(u + \lambda l \phi)\,dx\,ds. \tag{3.16}$$

Multiplying the second equation in $(P6)$ by $\Delta\phi$ and integrating over $Q_t$, $t \in (0,T]$, and using Green's formula, Fubini's theorem, and (1.5) (which guarantees the summability of the terms containing the Nemytski operators $k(x,s,\phi)$ and $h(\phi)$), we arrive at the equality

$$\frac{1}{2}\int_\Omega |\nabla\phi(x,t)|^2 dx + \int_{Q_t} (\Delta\phi)^2\,dx\,ds$$

$$= \frac{1}{2}\int_\Omega |\nabla \phi_0|^2 dx - \int_{Q_t} k(x,s,\phi)\Delta\phi\,dx\,ds$$

$$+ \int_{Q_t} h(\phi)\Delta\phi\,dx\,ds - \lambda\int_{Q_t} u\Delta\phi\,dx\,ds, \qquad \forall t \in (0,T]. \tag{3.17}$$

Since $\phi \in L^p(Q)$, we may use the chain rule given in formula (2.17) with $v = \phi$. From this, Green's formula, and (3.17) we derive

$$\frac{1}{2}\int_\Omega |\nabla\phi(x,t)|^2 dx + \int_{Q_t} (\Delta\phi)^2\,dx\,ds + \int_{Q_t} h'(\phi)|\nabla\phi|^2 dx\,ds$$

$$= \frac{1}{2}\int_\Omega |\nabla \phi_0|^2 dx - \int_{Q_t} k(x,s,\phi)\Delta\phi\,dx\,ds$$

$$- \lambda\int_{Q_t} u\Delta\phi\,dx\,ds,$$

$$\forall t \in (0,T]. \tag{3.18}$$

Adding (3.16) and (3.18) multiplied by $\frac{3}{2}(1 + \lambda^2 l^2)$, we obtain, by using Young's inequality, hypothesis $(H_3)$ (i), (3.14), and (3.15), the boundedness

$$\frac{1}{2}\int_\Omega \Big(|\nabla u + \lambda l \nabla\phi|^2(x,t) + \frac{3}{2}(1 + \lambda^2 l^2)|\nabla\phi|^2(x,t)\Big)dx$$

$$+ \int_{Q_t} \Big(\frac{1}{2}(\Delta u)^2 + \lambda l \Delta u \Delta\phi + \big(\frac{1}{2} + \lambda^2 l^2\big)(\Delta\phi)^2\Big)dx\,ds + \frac{3}{2}(1 + \lambda^2 l^2)$$

$$\int_{Q_t} h'(\phi)|\nabla\phi|^2 dx\,ds \le C(T, a_1, a_2), \qquad \forall t \in (0,T].$$

Now, from the inequality above, relation (3.14), and assumption $(H_3)$ (ii), we conclude

$$\int_{Q_t}\left[(\Delta u)^2 + (\Delta \phi)^2\right]dx\,ds \le C(T, a_1, a_2), \qquad \forall t \in (0, T]. \quad (3.19)$$

Finally, let us multiply the first equation in $(P6)$ by $\frac{\partial u}{\partial t} + \lambda l \frac{\partial \phi}{\partial t}$, integrate over $Q_t$, $t \in (0, T]$, and use Green's formula and Fubini's theorem to obtain

$$\int_{Q_t}\left(\frac{\partial u}{\partial t} + \lambda l \frac{\partial \phi}{\partial t}\right)^2 dx\,ds + \tfrac{1}{2}\int_{\Omega}|\nabla u(x,t)|^2 dx = \tfrac{1}{2}\int_{\Omega}|\nabla u_0(x)|^2 dx$$

$$+ \lambda l \int_{Q_t} \Delta u \frac{\partial \phi}{\partial t} dx\,dx + \lambda \int_{Q_t} f\left(\frac{\partial u}{\partial t} + \lambda l \frac{\partial \phi}{\partial t}\right) dx\,ds, \qquad \forall t \in (0, T]. \quad (3.20)$$

Multiplying the second equation in $(P6)$ by $\frac{\partial \phi}{\partial t}$ and then proceeding as in (3.20) leads to the equality

$$\int_{Q_t}\left(\frac{\partial \phi}{\partial t}\right)^2 dx\,ds + \frac{1}{2}\int_{\Omega}|\nabla \phi(x,t)|^2 dx$$

$$= \frac{1}{2}\int_{\Omega}|\nabla \phi_0(x)|^2 dx + \int_{Q_t}(k(x,t,\phi) - h(\phi))$$

$$\times \frac{\partial \phi}{\partial t} dx\,ds + \lambda \int_{Q_t} u \frac{\partial \phi}{\partial t}\, dx\,ds, \qquad \forall t \in (0, T].$$

We observe that

$$h(\phi)\frac{\partial \phi}{\partial t} = \frac{\partial}{\partial t}\left(\int_0^{\phi(x,t)} h(\tau)\,d\tau\right), \qquad (x,t) \in Q,$$

since $h \in C^1(\mathbb{R})$ and $\phi(x, \cdot) \in H^1(0, T; \mathbb{R})$. It results that

$$\int_{Q_t}\left(\frac{\partial \phi}{\partial t}\right)^2 dx\,ds + \frac{1}{2}\int_{\Omega}|\nabla \phi(x,t)|^2 dx + \int_{\Omega}\int_0^{\phi(x,t)} h(\tau)\,d\tau\,dx$$

$$= \frac{1}{2}\int_{\Omega}|\nabla \phi_0(x)|^2 dx + \int_{\Omega}\int_0^{\phi_0(x)} h(\tau)\,d\tau\,dx$$

$$+ \int_{Q_t} k(x,t,\phi)\frac{\partial \phi}{\partial t} dx\,ds + \lambda \int_{Q_t} u \frac{\partial \phi}{\partial t} dx\,ds, \quad \forall t \in (0, T]. \quad (3.21)$$

Adding (3.20) and (3.21) multiplied by $\frac{5}{4} + 4\lambda^2 l^2$ and then arguing on the basis of

$$\int_{Q_t}\left[\left(\frac{\partial u}{\partial t}\right)^2 + 2\lambda l \frac{\partial u}{\partial t}\frac{\partial \phi}{\partial t} + \lambda^2 l^2\left(\frac{\partial \phi}{\partial t}\right)^2 + \left(\tfrac{5}{4} + 4\lambda^2 l^2\right)\left(\frac{\partial \phi}{\partial t}\right)^2\right]dx\,ds$$
$$+\frac{1}{2}\int_\Omega |\nabla u(x,t)|^2 dx + \tfrac{1}{2}\left(\tfrac{5}{4} + 4\lambda^2 l^2\right)\int_\Omega |\nabla \phi(x,t)|^2 dx$$
$$+\left(\tfrac{5}{4} + 4\lambda^2 l^2\right)\int_\Omega\int_0^{\phi(x,t)} h(\tau)\,d\tau\,dx$$
$$\leq C + \tfrac{1}{2}\int_{Q_t}\left(\frac{\partial u}{\partial t}\right)^2 dx\,ds + (1 + \lambda^2 l^2)\int_{Q_t}\left(\frac{\partial \phi}{\partial t}\right)^2 dx\,ds, \qquad \forall t \in (0,T]).$$

this reduces to

$$\int_{Q_t}\left[\tfrac{1}{4}\left(\frac{\partial u}{\partial t}\right)^2 + \tfrac{1}{4}\left(\frac{\partial \phi}{\partial t}\right)^2 + \left(\tfrac{1}{2}\frac{\partial u}{\partial t} + 2\lambda l \frac{\partial \phi}{\partial t}\right)^2\right]dx\,ds$$
$$+\tfrac{1}{2}\int_\Omega |\nabla u(x,t)|^2 dx + \tfrac{A}{2}\int_\Omega |\nabla \phi(x,t)|^2 dx + A\int_\Omega\int_0^{\phi(x,t)} h(\tau)\,d\tau\,dx$$
$$\leq C(T, a_1, a_2), \qquad \forall t \in (0,T]. \tag{3.22}$$

We invoke assumption $(H_3)$ (ii) to deduce that

$$\int_0^z h(\tau)\,d\tau \geq -b_0(1 + z^2), \qquad \forall z \in \mathbb{R}. \tag{3.23}$$

Relations (3.22), (3.23), and (3.14) lead to the following estimate ($\forall t \in (0, T]$):

$$\int_{Q_t}\left(\frac{\partial u}{\partial t}\right)^2 dx\,ds + \int_{Q_t}\left(\frac{\partial \phi}{\partial t}\right)^2 dx\,ds + \int_\Omega\left(|\nabla u(x,t)|^2 + |\nabla \phi(x,t)|^2\right)dx$$
$$\leq C(T, a_1, a_2, b_0). \tag{3.24}$$

The $L_p$-theory applied to the first two equations in $(P6)$ as well as the estimates (3.24), (3.19) implies the existence of a positive constant $C$ such that

$$\|u\|_{W_2^{2,1}(Q)} \leq C(|\Omega|)\left(\|u_0\|_{W_\infty^2(\Omega)} + \lambda\left\|-l\frac{\partial\phi}{\partial t} + f\right\|_{L^2(Q)}\right)$$

$$\leq C(|\Omega|, T, a_1, a_2, b_0), \tag{3.25}$$

$$\|\phi\|_{W_2^{2,1}(Q)} \leq C(|\Omega|)\left(\|\phi_0\|_{W_\infty^2(\Omega)} + \left\|\frac{\partial\phi}{\partial t} - \Delta\phi\right\|_{L^2(Q)}\right)$$

$$\leq C(|\Omega|, T, a_1, a_2, b_0). \tag{3.26}$$

In view of continuous embedding (2.22), the a priori estimates (3.25), (3.26) ensure that a constant $\rho > 0$ can be found such that the property expressed in (3.9) holds. Q.E.D.

### 3.2. *Proof of Theorem* 1.1

Fix some $p$ satisfying (2.2). According to Lemma 3.2 we know the existence of a number $\rho > 0$ which fulfills the property stated in (3.9). Let us consider the open ball

$$B_\rho := \{(u, \phi) \in L^q(Q) \times L^p(Q) : \|(u, \phi)\|_{L^q(Q)\times L^p(Q)} < \rho\}.$$

Lemma 3.1 ensures that the mapping $H\colon L^q(Q) \times L^p(Q) \times [0,1] \to L^q(Q) \times L^p(Q)$ introduced in (3.1) is a homotopy of compact transformations on the closed ball $\overline{B}_\rho$ (see Fonseca and Gangbo [8, pp. 178–179]). Lemma 3.2 implies that

$$H(u, \phi, \lambda) \neq (u, \phi), \qquad \forall (u, \phi) \in \partial B_\rho, \quad \forall \lambda \in [0,1].$$

The foregoing properties allow us to consider the Leray–Schauder degree $\deg(Id - H(\cdot, \cdot, \lambda), B_\rho, 0)$, $\forall \lambda \in [0,1]$ (see Fonseca and Gangbo [8, Definition 7.6]). The homotopy invariance of Leray–Schauder degree shows that the equality below holds

$$\deg\big(Id - H(\cdot, \cdot, 0), B_\rho, 0\big) = \deg\big(Id - H(\cdot, \cdot, 1), B_\rho, 0\big). \tag{3.27}$$

We note that the equality $(u, \phi) = H(u, \phi, 0)$ is equivalent to the following nonlinear system of decoupled parabolic equations:

$$(P7) \qquad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = 0 & \text{in } Q, \\ \dfrac{\partial \phi}{\partial t} - \Delta \phi = F(x, t, \phi) & \text{in } Q, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial \phi}{\partial \nu} = 0 & \text{on } \Sigma, \\ u(x, 0) = u_0(x), \quad \phi(x, 0) = \phi_0(x) & x \in \Omega. \end{cases}$$

Theorems 2.1 and 2.2 provide a unique solution $\phi \in W_q^{2,1}(Q)$ of the second equation in $(P7)$. We can choose a sufficiently large $\rho > 0$ such that the ball $B_\rho$ contains the unique solution $(u, \phi)$ of $(P7)$. Taking into account that $H(\cdot, \cdot, 0)$ is a constant map, it turns out that $\deg(Id - H(\cdot, \cdot, 0), B_\rho, 0) = 1$. Then relation (3.27) ensures that the equation $(u, \phi) - H(u, \phi, 1) = 0$ has a solution $(u, \phi) \in B_\rho \subset L^q(Q) \times L^p(Q)$. By (3.1) and $(P5)$ with $\lambda = 1$, this is just a solution of problem $(P)$. The regularity part in Theorem 2.1 shows that in fact, $\phi \in W_\mu^{2,1}(Q)$, with $\mu$ given in (1.6).

Now we prove the estimate (1.7). Using Theorem 2.1 for the second equation in problem $(P)$, we get

$$\begin{aligned} \|\phi\|_{W_q^{2,1}(Q)} \le C\big(&|\Omega|, T, q, r, a_1, a_2, b_0, c_0, \|F(\cdot, \cdot, 0)\|_{L^\infty(Q)}\big) \\ &\times\big(1 + \|\phi_0\|^r_{W^2_\infty(\Omega)} + \|u\|_{L^q(Q)}\big). \end{aligned} \tag{3.28}$$

The $L_p$-theory applied to the first equation in $(P)$, combined with the relations (3.28), (2.22), implies the estimate

$$\begin{aligned} \|u\|_{W_q^{2,1}(Q)} &\le C\left(\|u_0\|_{W^2_\infty(\Omega)} + \left\|\frac{\partial \phi}{\partial t}\right\|_{L^q(Q)} + \|f\|_{L^q(Q)}\right) \\ &\le C\big(1 + \|u_0\|_{W^2_\infty(\Omega)} + \|\phi_0\|^r_{W^2_\infty(\Omega)} + \|u\|_{L^q(Q)} + \|f\|_{L^q(Q)}\big) \\ &\le C\big(1 + \|u_0\|_{W^2_\infty(\Omega)} + \|\phi_0\|^r_{W^2_\infty(\Omega)} + \|u\|_{W_2^{2,1}(Q)} + \|f\|_{L^q(Q)}\big). \end{aligned} \tag{3.29}$$

We note that (3.25) expresses that $\|u\|_{W_2^{2,1}(Q)} \le C(|\Omega|, T, q, r, a_1, a_2, b_0)$ $(1 + \|u_0\|_{W^2_\infty(\Omega)} + \|f\|_{L^2(Q)})$. Therefore, relations (3.28), (3.29) lead to estimate (1.7).

Denote $u = u_1 - u_2$, $\phi = \phi_1 - \phi_2$ and $f = f_1 - f_2$. We know that $u \in W_q^{2,1}(Q)$ and $\phi \in W_\mu^{2,1}(Q)$ verify problem $(P)$ with $u_0 = \phi_0 = 0$. Using Theorem 2.2 we find

$$\begin{aligned} \|\phi_1 - \phi_2\|_{W_\mu^{2,1}(Q)} = \|\phi\|_{W_\mu^{2,1}(Q)} &\le C\big(|\Omega|, T, q, r, a_0, M\big)\|u\|_{L^\mu(Q)} \\ &= C\big(|\Omega|, T, q, r, a_0, M\big)\|u_1 - u_2\|_{L^\mu(Q)}. \end{aligned} \tag{3.30}$$

The $L_p$-theory applied to the first equation in $(P)$ combined with (3.30) shows that

$$\begin{aligned} \|u_1 - u_2\|_{W_q^{2,1}(Q)} &= \|u\|_{W_q^{2,1}(Q)} \\ &\le C\big(|\Omega|, T, q, r, a_0, M\big)\big(\|u_1 - u_2\|_{L^\mu(Q)} + \|f_1 - f_2\|_{L^q(Q)}\big). \end{aligned} \tag{3.31}$$

The following interpolation inequality holds (see (2.33′)):

$$\|u_1 - u_2\|_{L^{\mu}(Q)} \leq \varepsilon \|u_1 - u_2\|_{W_q^{2,1}(Q)} + C(\varepsilon)\|u_1 - u_2\|_{L^2(Q)}.$$

Taking $\varepsilon > 0$ small enough, we deduce from (3.14), (3.30), (3.31) that estimate (1.8) holds.

The uniqueness of solution $(u, \phi)$ follows from relation (1.8) by taking $f_1 = f_2$. Q.E.D.

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## REFERENCES

1. R. A. Adams, "Sobolev spaces," Academic Press, Orlando/San Diego/New York, 1975.
2. H. Brézis, "Analyse Fonctionnelle. Théorie et Applications," Masson, Paris, 1983.
3. P. W. Bates and P. C. Fife, Spectral comparison principles for the Cahn–Hilliard and phase-field equations, and time scales for coarsening, *Phys. D.* **43** (1990), 335–348.
4. P. W. Bates and S. Zheng, Inertial manifolds and inertial sets for the phase-field equations, *J. Dynam. Differential Equations* **4** (1992), 375–398.
5. G. Caginalp, An analysis of a phase field model of a free boundary, *Arch. Rational Mech. Anal.* **92** (1986), 205–245.
6. C. M. Elliot and S. Zheng, Global existence and stability of solutions to the phase field equations, in "Free Boundary Problems" (K.-H. Hoffman and J. Sprekels, Eds.), pp. 46–58, Birkhäuser, Basel, 1990.
7. P. C. Fife, Models for phase separation and their mathematics, in "Nonlinear Partial Differential Equations and Applications" (M. Mimura and Y. Nishiura, Eds.), Kinokuniya, Tokyo, to appear.
8. I. Fonseca and W. Gangbo, "Degree Theory in Analysis and Applications," Clarendon, Oxford, 1995.
9. K.-H. Hoffman and L. Jiang, Optimal control of a phase field model for solidification, *Numer. Funct. Anal. Optim.* **13** (1992), 11–27.
10. O. A. Ladyzhenskaya, B. A. Solonnikon, and N. N. Uraltzeva, "Linear and Quasi-linear Equations of Parabolic Type," Amer. Math. Soc., Providence, 1968.
11. J. L. Lions, "Control of distributed singular systems," Gauthier-Villars, Paris, 1985.
12. C. Moroşanu and D. Motreanu, An extension of the nonlinear equation in Caginalp's model, *Comm. Appl. Anal.* **2** (1998), 159–168.
13. C. Moroşanu and D. Motreanu, Uniqueness and approximation for the nonlinear parabolic equation in Caginalp's model, ZAMM, *Z. Angew. Math. Mech.*, to appear.
14. O. Penrose and P. C. Fife, Thermodynamically consistent models of phase-field type for kinetics of phase transitions, *Phys. D.* **43** (1990), 44–62.
15. J. T. Schwartz, "Nonlinear functional analysis," Gordon and Breach, New York, 1969.